Generalization of the Gell-Mann formula for $sl(5, \mathbb{R})$ and $su(5)$ algebras *

Igor Salom †

Institute of Physics, P.O. Box 57, 11001 Belgrade, Serbia

Djordje Šijački ‡

Institute of Physics, P.O. Box 57, 11001 Belgrade, Serbia

June 11, 2009

Abstract

The so called Gell-Mann formula expresses the Lie algebra elements in terms of the corresponding Inönü-Wigner contracted ones. In the case of $sl(n, \mathbb{R})$ and $su(n)$ algebras contracted w.r.t. $so(n)$ subalgebras, the Gell-Mann formula is generally not valid, and applies only in the cases of some algebra representations. A generalization of the Gell-Mann formula for $sl(5, \mathbb{R})$ and $su(5)$ algebras, that is valid for all representations, is obtained in a group manifold framework of the $SO(5)$ and/or $Spin(5)$ group.

PACS: 02.20.Sv, 02.20.Qs; MSC2000: 20C33, 20C40;

1 Introduction

The Gell-Mann, or "decontraction" formula is a simple prescription designed to determine a deformation of a Lie algebra that is "inverse" to the Inönü-Wigner contraction [1]. This formula expresses elements of "decontracted"

* This work was supported in part by MNTR, Belgrade, Project-141036.

† e-mail address: isalom@phy.bg.ac.rs

‡ e-mail address: sijacki@phy.bg.ac.rs

algebra in terms of elements of the contracted one. Since, by a rule, various properties of the contracted algebra are much easier to explore (e.g. construction of representations [2], decompositions of a direct product of representations [3], etc.), this formula found its place, as a useful and simple tool, even in some textbooks [3] and in the mathematical encyclopedia [4].

There is a myriad of Inönü-Wigner Lie algebra contraction applications arising in various parts of Theoretical Physics. Just to mention a few ranging from contractions from the Poincaré algebra to the Galilean one, and from the Heisenberg algebras to the Abelian ones of the same dimensions (a symmetry background of a transition processes from relativistic and quantum mechanics to classical mechanics) to those of contractions from (Anti-)deSitter to the Poincaré algebra, and various cases involving Virasoro and Kac-Moody algebras. A recent study of an Affine Gauge Gravity Theory in $5D$ [5] is heavily related to the $sl(5,\mathbb{R})$ algebra contraction w.r.t. its $so(1,3)$ subalgebra, and the representations of the relevant algebras.

The main drawback of the Gell-Mann formula is its limited validity. There is a number of references dealing with the question when this formula is applicable [3, 6, 7]. The formula is best studied in the case of (pseudo) orthogonal algebras $so(m,n)$ contracted w.r.t. their $so(m-1,n)$ and/or $so(m,n-1)$ subalgebras, i.e. on the group level for $SO(m,n) \rightarrow R^{m+n-1} \wedge SO(m-1,n)$ or $SO(m,n) \rightarrow R^{m+n-1} \wedge SO(m,n-1)$, where, loosely speaking, the Gell-Mann formula works very well [8]. Moreover, the case of (pseudo) orthogonal algebras is the only one where this formula is valid for (almost) all representations [9]. There were some attempts to generalize the Gell-Mann formula [10, 11], that resulted in a construction of relatively complicated polynomial formulas for the "decontracted" algebra operators, applicable to complex simple Lie algebras g with respect to decomposition $g = k + ik = k_c$.

In this work we are generally interested in Gell-Mann's formula for the $sl(n,\mathbb{R})$ algebras, that are contracted w.r.t. their maximal compact $so(n)$ subalgebras. Note that, due to mutual relations of the $sl(n,\mathbb{R})$ and $su(n)$ algebras, one can convey the results obtained for the $sl(n,\mathbb{R})$ algebras to the corresponding ones of the $su(n)$ algebras. There are some subtleties in that process that will be considered below. The Gell-Mann formula is, in this case, especially valuable for the problem of finding all unitary irreducible representations of the $sl(n,\mathbb{R})$ algebras in the basis of the $SO(n)$ and/or $Spin(n)$ groups generated by their $so(n)$ subalgebras. Finding representations in the basis of the maximal compact subgroup $SO(n)$ of the $SL(n,\mathbb{R})$ group, is mathematically superior, and it suites well various physical applications in

particular in nuclear physics, gravity, physics of p-branes [12] etc. Moreover, this framework opens up a possibility of finding, in a rather straightforward manner, all matrix elements of noncompact $SL(n,\mathbb{R})$ generators for all representations. Unfortunately, the original Gell-Mann formula is, in that respect, limited to some classes of (multiplicity free) representations only.

As already stated, the Gell-Mann formula, except in the cases of (pseudo) orthogonal algebras, is not generally valid by itself, and its validity depends on the representations of the algebra as well. Therefore, in the case of the $SL(n,\mathbb{R})$ groups, i.e. their $sl(n,\mathbb{R})$ algebras, one is faced, in addition to the pure algebraic features, with a problem of studying the matters that are relevant to characterize representations as well: (i) the group topology properties, and (ii) the non trivial multiplicity of the $SL(n,\mathbb{R})$, and $sl(n,\mathbb{R})$ representations in the $SO(n)$, and $so(n)$ basis, respectively. Both features are rather subtle for $n \geq 3$. Note that, in the case of the $sl(n,\mathbb{R})$ algebras, due to a fact that the generalization of the Gell-Mann formula obtained below depends on the algebra representation features, we deviate from the standard Lie algebra deformation approach.

The $SL(n,\mathbb{R})$ group can be decomposed, as any semisimple Lie group, into the product of its maximal compact subgroup $K = SO(n)$, an Abelian group A and a nilpotent group N. It is well known that only K is not guaranteed to be simply-connected. There exists a universal covering group $\overline{K} = \overline{SO}(n)$ of $K = SO(n)$, and thus also a universal covering of $G = SL(n,\mathbb{R})$: $\overline{SL}(n,\mathbb{R}) \simeq \overline{SO}(n) \times A \times N$. For $n \geq 3$, $SL(n,\mathbb{R})$ has double covering, defined by $\overline{SO}(n) \simeq Spin(n)$ the double-covering of the $SO(n)$ subgroup. The universal covering group $\overline{G}$ of a given group G is a group with the same Lie algebra and with a simply-connected group manifold. A finite dimensional covering, $\overline{SL}(n,\mathbb{R})$ exists provided one can embed $\overline{SL}(n,\mathbb{R})$ into a group of finite complex matrices that contain $Spin(n)$ as subgroup. A scan of the Cartan classical algebras points to the $SL(n,C)$ groups as a natural candidate for the $SL(n,\mathbb{R})$ groups covering. However, there is no match of the defining dimensionalities of the $SL(n,C)$ and $Spin(n)$ groups for $n \geq 3$, $dim(SL(n,C)) = n < 2^{\left[\frac{n-1}{2}\right]} = dim(Spin(n))$, except for $n = 8$. In the $n = 8$ case, one finds that the orthogonal subgroup of the $SL(8,\mathbb{R})$ and $SL(8,C)$ groups is $SO(8)$ and not $Spin(8)$. For a detailed account of the $D = 4$ case cf. [13]. Thus, we conclude that there are no covering groups of the $SL(n,\mathbb{R})$, $n \geq 3$ groups defined in finite-dimensional spaces. An explicit construction of all $SL(3,R)$ irreducible representations, unitary and

nonunitary multiplicity-free spinorial [14], and unitary non-multiplicity-free [15], shows that they are infinite-dimensional. The universal (double) covering groups, $\overline{SL}(n,R)$, $n \geq 3$ of the $SL(n,R)$, $n \geq 3$ group are groups of infinite complex matrices. All their spinorial representations are infinite dimensional. In the reduction of this representations w.r.t. $Spin(n)$ subgroups, one finds $Spin(n)$ representations of unbounded spin values.

The $SU(n)$ groups are compact, with a simply connected group manifold, thus being its own universal coverings. The $SO(n)$ subgroups are embedded into the $SU(n)$ groups as n-dimensional matrices, and this embedding does not allow nontrivial (double) covering of $SO(n)$ within $SU(n)$. As a consequence, in the reduction of the $SU(n)$ unitary irreducible representations one finds the tensorial $SO(n)$ representations only.

An inspection of the unitary irreducible representations of the $\overline{SL}(n,\mathbb{R})$, $n = 3,4$ groups [15, 16] shows that they have, as a rule, a nontrivial multiplicity of the $Spin(n)$, $n = 3,4$ subgroup representations. It is well known, already from the case of the $SU(3)$ representations in the $SO(3)$ subgroup basis, that the additional labels required to describe this nontrivial multiplicity cannot be solely related to the group generators themselves. An elegant solution, that provides the required additional labels, is to work in the group manifold of the $SO(n)$ maximal compact subgroup, and to consider an action of the group both to the right and to the left. In this way one obtains, besides the maximal compact subgroup labels, an additional set of labels to describe the $SO(n)$ subgroup multiplicity.

All unitary irreducible representations of the $\overline{SL}(3,\mathbb{R})$ and $\overline{SL}(4,\mathbb{R})$ groups are classified, and various relevant explicit expressions are known [15, 16]. It turns out that an answer to the question of the Gell-Mann formula generalization can be effectively read out from these known closed form expressions of representations of noncompact generators in both $SL(3,\mathbb{R})$ and $SL(4,\mathbb{R})$ cases. Such a generalization then, as a rule, has an overall validity for all representations. We study the known representations of the noncompact $SL(3,\mathbb{R})$ and $SL(4,\mathbb{R})$ generators in the maximal compact subgroup basis, and infer the sought for expressions for the corresponding Gell-Mann formula. On the basis of these results, we turn to the case of the $SL(5,\mathbb{R})$ generators, whose unitary irreducible representations are not known completely. As a result, we obtain a single closed expression that generalizes the Gell-Mann formula for the $sl(5,\mathbb{R})$ algebra w.r.t. its maximal compact $so(5)$ subalgebra.

2 Inönü-Wigner contraction of $sl(n, R)$ algebras

The $sl(n,\mathbb{R})$ algebra operators, i.e. the $SL(n,\mathbb{R})$ group generators, can be split into two subsets: M_{ab}, $a,b = 1,2,...,n$ operators of the maximal compact subalgebra $so(n)$ (corresponding to the antisymmetric real $n \times n$ matrices, $M_{ab} = -M_{ba}$), and the, so called, sheer operators T_{ab}, $a,b = 1,2,...,n$ (corresponding to the symmetric traceless real $n \times n$ matrices, $T_{ab} = T_{ba}$). The $sl(n,\mathbb{R})$ commutation relations, in this basis, read:

$$\begin{aligned}
[M_{ab}, M_{cd}] &= i(\delta_{ac}M_{bd} + \delta_{ad}M_{cb} - \delta_{bc}M_{ad} - \delta_{bd}M_{ca}), && (1)\\
[M_{ab}, T_{cd}] &= i(\delta_{ac}T_{bd} + \delta_{ad}T_{cb} - \delta_{bc}T_{ad} - \delta_{bd}T_{ca}), && (2)\\
[T_{ab}, T_{cd}] &= i(\delta_{ac}M_{db} + \delta_{ad}M_{cb} + \delta_{bc}M_{da} + \delta_{bd}M_{ca}). && (3)
\end{aligned}$$

The $su(n)$ algebra operators can be split likewise w.r.t. its $so(n)$ subalgebra into M_{ab} and $T^{su(n)}_{ab}$, $a,b = 1,2,...,n$. The $T^{su(n)}_{ab}$ and T_{ab} operators are mutually related by $T^{su(n)}_{ab} = \mathrm{i}\ T_{ab}$, and the $[T^{su(n)}_{ab}, T^{su(n)}_{cd}]$ differs from (3) by having an overall plus sign on the right-hand side.

The Inönü-Wigner contraction of $sl(n,\mathbb{R})$ with respect to its maximal compact subalgebra $so(n)$ is given by the limiting procedure:

$$U_{ab} \equiv \lim_{\epsilon \to 0}(\epsilon T_{ab}), \qquad (4)$$

which leads to the following commutation relations:

$$\begin{aligned}
[M_{ab}, M_{cd}] &= i(\delta_{ac}M_{bd} + \delta_{ad}M_{cb} - \delta_{bc}M_{ad} - \delta_{bd}M_{ca}) && (5)\\
[M_{ab}, U_{cd}] &= i(\delta_{ac}U_{bd} + \delta_{ad}U_{cb} - \delta_{bc}U_{ad} - \delta_{bd}U_{ca}) && (6)\\
[U_{ab}, U_{cd}] &= 0. && (7)
\end{aligned}$$

Therefore, the Inönü-Wigner contraction of $sl(n,\mathbb{R})$ gives a semidirect sum $r_{\frac{n(n+1)}{2}-1} \uplus so(n)$ algebra, where $r_{\frac{n(n+1)}{2}-1}$ is an Abelian subalgebra (ideal) of "translations" in $\frac{n(n+1)}{2} - 1$ dimensions.

The Gell-Mann formula, which is a prescription to provide an "inverse" to the contraction, (4), in this case reads:

$$T_{ab} = \sigma U_{ab} + \frac{i\alpha}{\sqrt{U \cdot U}}[C_2(so(n)), U_{ab}], \qquad (8)$$

where $C_2(so(n))$ denotes the second order Casimir operator of the $so(n)$ algebra, $\frac{1}{2}\sum M_{ab}M_{ab}$, while σ is an arbitrary (complex) parameter and α is a (real) normalization constant that depends on n.

In order to make use of the Gell-Mann formula to obtain $sl(n,\mathbb{R})$ representations, the first necessary step is to construct representations of the contracted algebra. Representations of the $so(n)$ generators M_{ab} are well known. There are two properties that characterize representations of the U_{ab} operators: (i) The U_{ab} operators transform w.r.t the $\frac{n(n+1)}{2}-1$ dimensional representation of $so(n)$ (6), i.e. as a symmetric second order $so(n)$ tensors (in Young diagram notation $\square\!\square$), and (ii) U_{ab} mutually commute. These two requirements are met by expressing the U_{ab} operators in terms of the, so called, Wigner D-function (the $SO(n)$ group matrix elements expressed as functions of the group parameters):

$$U_{ab} \sim D^{(\square\!\square)}_{(cd)(ab)}(g^{-1}(\theta)) \equiv \left\langle \begin{array}{c}\square\!\square\\(cd)\end{array} \right| g^{-1}(\theta) \left| \begin{array}{c}\square\!\square\\(ab)\end{array} \right\rangle, \tag{9}$$

$g(\theta)$ being an $SO(n)$ element parameterized by θ, pairs of indices (ab) and (cd) label the $SO(n)$ matrix elements, while $\left| \begin{array}{c}\square\!\square\\(ab)\end{array} \right\rangle$ form a basis of the $\square\!\square$ representation space. Taking inverse of g in (9) insures the correct transformation properties.

The contracted $r_{\frac{n(n+1)}{2}-1} \uplus so(n)$ and $sl(n,\mathbb{R})$ algebras are represented in a space of square integrable functions over the $Spin(n)$ group (in accord with the $SL(n,\mathbb{R})$ topological properties), with a standard invariant Haar measure: $\mathcal{L}^2(Spin(n))$. Harish-Chandra proved [17] that this space is rich enough to contain all possible representations (up to equivalence) of the $\overline{SL}(n,\mathbb{R})$ group, i.e. $sl(n,\mathbb{R})$ algebra. The U_{ab} operators act multiplicatively on this space, while the $so(n)$ subalgebra operators act, in a standard way, via a left group action:

$$M_{ab}\left|\phi\right\rangle = -i\frac{d}{dt}\exp(itM_{ab})\Big|_{t=0}\left|\phi\right\rangle, \quad g'\left|g\right\rangle = \left|g'g\right\rangle, \quad \left|\phi\right\rangle \in \mathcal{L}^2(Spin(n)).$$

This representation space is highly reducible, however this fact is of no relevance for the present considerations.

The U_{ab} expressions (9) fulfill, straightforwardly, both required properties: commute mutually, as being ordinary functions of θ, and transform under $\square\!\square$ when acting to the right on a ket vector $\left| \begin{array}{c}\square\!\square\\(ab)\end{array} \right\rangle$ (characterized accordingly

by the (ab) indices). However, the bra vector $\left\langle \begin{matrix} \square\square \\ (cd) \end{matrix} \right|$, standing to the left of $R(\theta)$, can be an arbitrary vector from the $\square\square$ representation, thus providing an additional set of labels of the U_{ab}, and accordingly the T_{ab}, operators. It turns out, as it will be seen below, that these additional labels play an important role in the $sl(n,\mathbb{R})$ unitary irreducible representations description, in particular in characterizing nontrivial $Spin(n)$ subgroup multiplicity.

A natural orthonormal basis in the $Spin(n)$ representation space is given by properly normalized functions of the $Spin(n)$ representation matrix elements:

$$\left\{ \left| \begin{matrix} J \\ km \end{matrix} \right\rangle \equiv \int \sqrt{dim(J)} D^J_{km}(g(\theta)^{-1}) d\theta \, |g(\theta)\rangle \right\}, \; \left\langle \begin{matrix} J \\ km \end{matrix} \middle| \begin{matrix} J' \\ k'm' \end{matrix} \right\rangle = \delta_{JJ'} \delta_{kk'} \delta_{mm'}, \tag{10}$$

where $d\theta$ is an (normalized) invariant Haar measure, and D^J_{km} are the representation matrix elements

$$D^J_{km}(\theta) \equiv \left\langle \begin{matrix} J \\ k \end{matrix} \middle| R(\theta) \middle| \begin{matrix} J \\ m \end{matrix} \right\rangle .$$

Here, J stands for a set of $Spin(n)$ irreducible representation labels, while k and m labels numerate representation basis vectors.

An action of the $so(n)$ operators in this basis is well known, and it can be written in terms of the Clebsch-Gordan coefficients of the $Spin(n)$ group as follows,

$$\langle M_{ab} \rangle = \left\langle \begin{matrix} J' \\ k'm' \end{matrix} \middle| M_{ab} \middle| \begin{matrix} J \\ km \end{matrix} \right\rangle = \delta_{JJ'} \sqrt{C_2(J)} \; C^{J \substack{\square \\ \square} J'}_{m (ab) m'} \; . \tag{11}$$

The matrix elements of the U_{ab} operators in this basis are readily found to read:

$$\begin{aligned} \langle U_{ab} \rangle &= \left\langle \begin{matrix} J' \\ k'm' \end{matrix} \middle| D^{-1 \square\square}_{(cd)(ab)} \middle| \begin{matrix} J \\ km \end{matrix} \right\rangle \\ &= \sqrt{dim(J')dim(J)} \int D^{J'*}_{k'm'}(\theta) D^{\square\square}_{(cd)(ab)}(\theta) D^J_{km}(\theta) d\theta \\ &= \sqrt{\tfrac{dim(J)}{dim(J')}} C^{J \square\square J'}_{k (cd) k'} \; C^{J \square\square J'}_{m (ab) m'} \end{aligned} \tag{12}$$

A closed form of the matrix elements of the whole contracted algebra $r_{\frac{n(n+1)}{2}-1} \uplus so(n)$ representations is thus explicitly given in this space by (11) and (12).

3 The Gell-Mann formula for the $sl(3,\mathbb{R})$ and $sl(4,\mathbb{R})$ algebras

Let us now, equipped with the knowledge about the contracted algebra representation, consider the Gell-Mann formula in the cases of the $sl(3,\mathbb{R})$ and $sl(4,\mathbb{R})$ algebras.

In the $sl(3,\mathbb{R})$ algebra case, the maximal compact subgroup of the corresponding $\overline{SL}(3,\mathbb{R})$ group is $Spin(3)$, and a basis of the $sl(3,\mathbb{R})$ representation space is given by the well known set of vectors,

$$\left\{ \left| \begin{array}{c} J \\ k\, m \end{array} \right\rangle, J = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots; \quad |k|, |m| \leq J \right\}. \tag{13}$$

The traceless symmetric tensor $\square\!\square$ transforms under a five-dimensional $SO(3)$, i.e. $Spin(3)$, representation of $J = 2$. One can make use of an arbitrary vector from this representation to evaluate the U_{ab} operators expressions. We take the simplest, however for our purposes adequate, realization of the U_{ab} operators, $U_{ab} \sim D^{(\square\!\square)}_{(33)(ab)}$, $a, b = 1, 2, 3$, i.e. in the spherical $SO(3)$ basis, $U_\mu \sim D^{(\square\!\square)}_{0\mu}$, $\mu = 0, \pm 1, \pm 2$. The Gell-Mann formula (8) yields now:

$$T_\mu = \sigma D^2_{0\mu} + i\alpha [C_2(so(3)), D^2_{0\mu}], \quad \mu = 0, \pm 1, \pm 2 \tag{14}$$

The matrix elements of the shear operators T_μ are given by the following expression:

$$\left\langle \begin{array}{c} J' \\ k'm' \end{array} \right| T_\mu \left| \begin{array}{c} J \\ k\,m \end{array} \right\rangle = \left(\sigma + i\alpha (J'(J'+1) - J(J+1))\right) \sqrt{\tfrac{2J+1}{2J'+1}} C^{J2J'}_{k0k'} \; C^{J\,2\,J'}_{m\mu m'} \tag{15}$$

The shear operators (14) satisfy the relation (2) by a construction. However the relation (3) is not a priory granted, and it must be checked to hold. An explicit calculation shows that this relation does not hold in general. It turns out, that the commutation relations of the $sl(3,\mathbb{R})$ algebra, as realized by (11) and (15), hold only for the representation subspaces characterized by $k = 0$, and provided $\alpha = 1/\sqrt{6}$, i.e. for

$$\left\{ \left| \begin{array}{c} J \\ 0\, m \end{array} \right\rangle, \quad J = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \quad |m| \leq J \right\}, \tag{16}$$

Therefore, the Gell-Mann formula is valid only for the $sl(3,\mathbb{R})$ representations defined in the Hilbert spaces over the $SO(3)/SO(2)$ coset space. These

representations are the multiplicity free ones w.r.t. the compact $so(3)$ subalgebra, since the label k, which counts the $so(3)$ representations multiplicity (i.e. in the physical terms the spin J degeneracy) is fixed.

The Gell-Mann prescription (14) does not work in the general case; a comparison with the complete classification of the $sl(3,\mathbb{R})$ representations [15] reveals that the $sl(3,\mathbb{R})$ representations with nontrivial multiplicity, as well as the spinorial multiplicity free representations, cannot be obtained in this way ($k=0$ implies that J must take strictly integer values). Moreover, a detailed analysis shows that this conclusion hold irrespectively of the concrete choice $U_\mu \sim D^{\square\square}_{0\mu}$ that we have made.

In the $sl(4,\mathbb{R})$ algebra case, the maximal compact subgroup of the corresponding $\overline{SL}(4,\mathbb{R})$ group, is $Spin(4)$. One possibility of choosing a basis for the $so(4)$ representations space corresponds to the $so(4)$ algebra decomposition $so(4) = so(3) \oplus so(3)$. The $so(4)$ representation space basis is now $\left\{ \left| \begin{array}{l} J_1\ J_2 \\ m_1 m_2 \end{array} \right\rangle \right\}$, where pairs (J_1, m_1) and (J_2, m_2) specify vectors of the two $SO(3)$ representations. A basis of the $sl(4,\mathbb{R})$ representation space (10) is then given by vectors

$$\left\{ \left| \begin{array}{l} J_1 J_2 \\ k_1 k_2 m_1 m_2 \end{array} \right\rangle, \quad J_i = 0, \frac{1}{2}, \ldots, \quad |k_i|, |m_i| \leq J_i, \quad i = 1,2 \right\}. \qquad (17)$$

Similarly, as in the $sl(3)$ case, the Gell-Mann formula yields:

$$T_{\mu_1\mu_2} = \sigma D^{11}_{00\mu_1\mu_2} + \frac{i}{2}[C_2(so(4)), D^{11}_{00\mu_1\mu_2}], \quad \mu = 0, \pm 1;\ i = 1,2. \qquad (18)$$

The corresponding matrix elements of the $T_{\mu_1\mu_2}$ operators read:

$$\begin{aligned} &\left\langle \begin{array}{l} J'_1 J'_2 \\ k'_1 k'_2 m'_1 m'_2 \end{array} \right| T_{\mu_1\mu_2} \left| \begin{array}{l} J_1 J_2 \\ k_1 k_2 m_1 m_2 \end{array} \right\rangle \\ &= \big(\sigma + i(J'_1(J'_1+1) + J'_2(J'_2+1) - J_1(J_1+1) - J_2(J_2+1))\big) \sqrt{\tfrac{(2J_1+1)(2J_2+1)}{(2J'_1+1)(2J'_2+1)}} \times \\ &\quad C^{J_1 1 J'_1}_{k_1 0 k'_1}\, C^{J_2 1 J'_2}_{k_2 0 k'_2}\, C^{J_1\ 1\ J'_1}_{m_1 \mu_1 m'_1}\, C^{J_2\ 1\ J'_2}_{m_2 \mu_2 m'_2}\ . \end{aligned} \qquad (19)$$

However, analogously as in the $sl(3,\mathbb{R})$ case, the commutation relations of the noncompact operators close correctly only for the subspaces (17) characterized by $k_1 = k_2 = 0$, and thus for J_1, J_2 being integers. The corresponding $sl(4,\mathbb{R})$ representations are the multiplicity free ones, while

the representation spaces are Hilbert spaces over the $(Spin(3)/Spin(2)) \times (Spin(3)/Spin(2))$ coset space ($Spin(2)$ denoting the double cover of $SO(2)$).

Additional $sl(4,\mathbb{R})$ multiplicity free representations can be obtained by working in the $Spin(4)$ representation spaces characterized by the subgroup chain $Spin(4) \supset Spin(3) \supset Spin(2)$, with basis vectors:

$$\left\{ \left| \begin{array}{c} J_1 J_2 \\ J \\ m \end{array} \right\rangle, \ J_i = 0, \frac{1}{2}, \dots; \ |J_1 - J_2| \leq J \leq J_1 + J_2; \ |m| \leq J; \ i = 1,2 \right\}. \tag{20}$$

The $sl(4,\mathbb{R})$ representation space is then defined by basis vectors:

$$\left\{ \left| \begin{array}{c} J_1 J_2 \\ K \ \ J \\ k \ \ m \end{array} \right\rangle, J_i = 0, \frac{1}{2}, \dots, |J_1 - J_2| \leq K, J \leq J_1 + J_2, |m| \leq J \right\}. \tag{21}$$

The corresponding Gell-Mann formula expression for the (noncompact) shear generators reads

$$T_{j\mu} = \sigma D^{\ 11}_{\begin{smallmatrix} 0 & j \\ 0 & \mu \end{smallmatrix}} - \tfrac{i\sqrt{3}}{4}[C_2(so(4)), D^{\ 11}_{\begin{smallmatrix} 0 & j \\ 0 & \mu \end{smallmatrix}}], \quad j = 0,1,2; \ |\mu| \leq j, \tag{22}$$

and yields the correct commutation relations in the subspace of (21) for $K = k = 0$, only. The corresponding $sl(4,\mathbb{R})$ representations are the multiplicity free ones, defined in symmetric spaces over the $Spin(4)/Spin(3)$ coset space. These representations have only the "diagonal" $so(4)$ content, since the condition $K = 0$ implies $J_1 = J_2$.

Note that, neither the representations with multiplicity, nor the spinorial multiplicity free $sl(4,\mathbb{R})$ representations (cf. [16]) can be obtained by making use of the expressions given by (18) or (22).

It is clear from these examples, that the Gell-Mann formula has a limited scope when applied to the $sl(n,\mathbb{R})$, $n = 3,4$ cases. It is neither valid as an operator expression, nor it holds for a generic $sl(n,\mathbb{R})$, $n = 3,4$ representation space. One can make use of the Gell-Mann formula in some subspaces of the most general representation space, only. More precisely, the expressions (14), (18) and (22) yield the noncompact algebra operators in the symmetric spaces over $Spin(3)/Spin(2)$, $Spin(4)/(Spin(2) \times Spin(2))$ and $Spin(4)/Spin(3)$, respectively. Note, that this result is in agreement with the Hermann theorem [6] stating that the Gell-Mann formula certainly works in the symmetric spaces K/L if K is a simple compact subgroup from a Cartan decomposition

of the starting group G (here $G = SL(n, \mathbb{R})$) and if there exists some U_μ which is invariant under the action of L. Unfortunately, this theorem does not give the necessary conditions for the Gell-Mann formula to hold.

It turns out that we can shed some light on the question of a validity of the Gell-Mann formula in the $sl(n, \mathbb{R})$, $n = 3, 4$ algebra case. In doing that, we recall first some relevant facts about the $SO(n)$, $n = 3, 4$ group manifold and its representations. Let us consider an action of the $SO(n)$ representation on the lower left-hand side labels (quantum numbers) of the basis vectors given by (10). First, let us make use of the operator

$$K_\mu \equiv g^{\nu\lambda} D^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{\mu\nu} M_\lambda, \quad \mu = 1, 2, \ldots, n \tag{23}$$

where $g^{\nu\lambda}$ is the Cartan metric tensor of $SO(n)$). The K_μ operators behave exactly as the rotation generators M_μ, it is only that they act on the lower left-hand side indices of the basis (10):

$$\langle K_{ab} \rangle = \left\langle \begin{smallmatrix} J' \\ k' m' \end{smallmatrix} \middle| K_{ab} \middle| \begin{smallmatrix} J \\ k m \end{smallmatrix} \right\rangle = \delta_{JJ'} \sqrt{C_2(J)}\, C^{J \begin{smallmatrix}\square\\\square\end{smallmatrix} J'}_{k\,(ab)\,k'} \ . \tag{24}$$

Due to the fact that mutually contragradient $SO(n)$ representations are equivalent, the K_μ operators are directly related to the "left" action of the $SO(n)$ subgroup on $\mathcal{L}^2(|g(\theta)\rangle)$: $g' \,|g\rangle = \left|g g'^{-1}\right\rangle$. The operators K_μ and M_μ mutually commute, however, the corresponding Casimir operators match, i.e. $K_\mu^2 = M_\mu^2$. Whereas the $sl(n, \mathbb{R})$ operators M_{ab} are invariant under this left action, the shear operators T_{ab}, constructed by using the Gell-Mann formula prescription, are not. The transformation properties of the shear operators T_{ab} are inherited from the corresponding contracted operators U_{ab}, which have additional nontrivial transformation properties described by the K_μ operator labels (of the right-hand side vector of the $SO(n)$ matrix in (9)). Consequently, a commutator of two such shear operators has a nontrivial properties under the $SO(n)_K$ group (generated by the K_μ operators). It is not an $SO(n)_K$ scalar (unlike M_μ), and therefore not bound to close upon the $sl(n, \mathbb{R})$ commutation relations. This is precisely the reason why the shear generators (14), (18) and (22) do not satisfy the commutation relations (3) in a generic representation space over the $SO(n)$ group manifold. In particular cases, it is possible to make a restriction to such subspace of the representation space, where only the invariant part of the $[T, T]$ commutator survives that is proportional to the $so(n)$ operators representation in that subspace. For example, in the $sl(3, \mathbb{R})$ case above, a restriction was made to

the subspace of $k = 0$, where the $[T, T]$ commutator piece proportional to K_0 vanishes; likewise for the commutators of the shear generators (18) and (22).

4 Generalization of the Gell-Mann formula for $sl(3, \mathbb{R})$ and $sl(4, \mathbb{R})$ algebras

The above analysis raises the question of whether it is possible to modify the Gell-Mann formula by adding some terms proportional to the generators of the left $SO(n)_K$ group, that cancel the unwanted terms and "fix" the $sl(n, \mathbb{R})$ algebra commutation relations.

Such a generalization of the Gell-Mann formula in the $sl(3, \mathbb{R})$ case can be read out directly from the known matrix elements of the $sl(3, \mathbb{R})$ representations with multiplicity [15]:

$$T_\mu = \sigma D^2_{0\mu} + \frac{i}{\sqrt{6}}[C_2(so(3)), D^2_{0\mu}] + i(D^2_{2\mu} - D^2_{-2\mu})K_0 + \delta(D^2_{2\mu} + D^2_{-2\mu}), \quad (25)$$

$\mu = 0, \pm 1, \pm 2$, and where σ and δ are the $sl(3, \mathbb{R})$ group representation labels. The additional terms to the "original" Gell-Mann formula secure that the T_μ operators satisfy the commutation relation (3) in the entire representation space. Note that, there are two $sl(3, \mathbb{R})$ representation labels σ and δ, matching the algebra rank, contrary to the case of the original Gell-Mann formula whose single free parameter cannot account for the entire representation labeling. The additional label k ($K_0 \to k$, $|k| \leq J$) describes the nontrivial multiplicity in J.

The generalized expression (25) contains the original formula (14) as a special case: by restricting the representation space to the subspace of $k = 0$, and choosing $\delta = 0$ one arrives at the multiplicity free representations that were obtained by using the expression (14). Moreover, the generalized Gell-Mann formula allows one to obtain some $sl(3, \mathbb{R})$ multiplicity free representations that cannot be reached by making use of the original formula (14). For example, the choice $\sigma = \frac{3}{2}$, and $\delta = -\frac{1}{2}$ [18] in a new basis of vectors (linear combinations of basis vectors with different k values),

$$\Big\{ \left| \begin{smallmatrix} \frac{1}{2} \\ m \end{smallmatrix} \right>' = \left| \begin{smallmatrix} \frac{1}{2} & \\ \frac{1}{2} & m \end{smallmatrix} \right> + \left| \begin{smallmatrix} \frac{1}{2} & \\ -\frac{1}{2} & m \end{smallmatrix} \right>,$$

$$\left| \begin{smallmatrix} \frac{5}{2} \\ m \end{smallmatrix} \right>' = \left| \begin{smallmatrix} \frac{5}{2} & \\ \frac{5}{2} & m \end{smallmatrix} \right> + \sqrt{\tfrac{5}{2}} \left| \begin{smallmatrix} \frac{5}{2} & \\ \frac{1}{2} & m \end{smallmatrix} \right> + \sqrt{\tfrac{5}{2}} \left| \begin{smallmatrix} \frac{5}{2} & \\ -\frac{1}{2} & m \end{smallmatrix} \right> + \left| \begin{smallmatrix} \frac{5}{2} & \\ -\frac{5}{2} & m \end{smallmatrix} \right>,$$

$$\left| \begin{smallmatrix} \frac{9}{2} \\ m \end{smallmatrix} \right>' = \left| \begin{smallmatrix} \frac{9}{2} & \\ \frac{9}{2} & m \end{smallmatrix} \right> + \left| \begin{smallmatrix} \frac{9}{2} & \\ \frac{5}{2} & m \end{smallmatrix} \right> + \sqrt{\tfrac{7}{2}} \left| \begin{smallmatrix} \frac{9}{2} & \\ \frac{1}{2} & m \end{smallmatrix} \right> + \sqrt{\tfrac{7}{2}} \left| \begin{smallmatrix} \frac{9}{2} & \\ -\frac{1}{2} & m \end{smallmatrix} \right> + \left| \begin{smallmatrix} \frac{9}{2} & \\ -\frac{5}{2} & m \end{smallmatrix} \right> + \left| \begin{smallmatrix} \frac{9}{2} & \\ -\frac{9}{2} & m \end{smallmatrix} \right>, \dots \Big\}.$$

yields a representation space without multiplicity that is closed under the action of the T_μ operators. This is a basis of a spinorial $sl(3,\mathbb{R})$ unitary irreducible (J content are half-odd integers) representation space, where the original Gell-Mann formula does not apply.

The Gell-Mann formula can similarly be generalized in the case of the $sl(4,\mathbb{R})$ algebra. Again, by extracting from the known matrix elements of the $sl(4,\mathbb{R})$ representations with multiplicity [16], we find:

$$
\begin{aligned}
T_{\mu_1\mu_2} = i\Big(&\sigma D^{11}_{00\mu_1\mu_2} + \tfrac{1}{2}[C_2(so(4)), D^{11}_{00\mu_1\mu_2}] \\
&+\delta_1(D^{11}_{11\mu_1\mu_2} + D^{11}_{-1-1\mu_1\mu_2}) + (D^{11}_{11\mu_1\mu_2} - D^{11}_{-1-1\mu_1\mu_2})(K^{10}_{00} + K^{01}_{00}) \\
&+\delta_2(D^{11}_{-11\mu_1\mu_2} + D^{11}_{1-1\mu_1\mu_2}) + (D^{11}_{-11\mu_1\mu_2} - D^{11}_{1-1\mu_1\mu_2})(K^{10}_{00} - K^{01}_{00})\Big),
\end{aligned} \tag{26}
$$

where $\mu_1, \mu_2 = 0, \pm 1$. As the rank of the $sl(4,\mathbb{R})$ algebra is three, there are precisely three representation labels σ, δ_1, and δ_2 (if complex, only three real are independent).

As in the $sl(3,\mathbb{R})$ case, the generalized formula reduces, for certain values of the labels, in a representation subspace defined by $k_1 = k_2 = 0$ to the original Gell-Mann formula (18). It is not a straightforward matter to see that the formula (22) follows from the generalized Gell-Mann formula. However, the generalized Gell-Mann formula for the $sl(4,\mathbb{R})$ algebra can be expressed in an equivalent form as follows:

$$
\begin{aligned}
T_{j\mu} = {}&\gamma_1 D^{\substack{11\\0\;j}}_{\substack{0\;\mu}} - \tfrac{i\sqrt{3}}{4}[C_2(so(4)), D^{\substack{11\\0\;j}}_{\substack{0\;\mu}}] \\
&+\gamma_2 D^{\substack{11\\2\;j}}_{\substack{0\;\mu}} + i\sqrt{2} D^{\substack{11\\2\;j}}_{\substack{1\;\mu}} (K^{\substack{10\\1}}_{\substack{-1}} + K^{\substack{01\\1}}_{\substack{-1}}) - i\sqrt{2} D^{\substack{11\\2\;j}}_{\substack{-1\mu}} (K^{\substack{10\\1}}_{\substack{1}} + K^{\substack{01\\1}}_{\substack{1}}) \\
&+\gamma_3(D^{\substack{11\\2\;j}}_{\substack{2\;\mu}} + D^{\substack{11\\2\;j}}_{\substack{-2\mu}}) + i(D^{\substack{11\\2\;j}}_{\substack{2\;\mu}} - D^{\substack{11\\2\;j}}_{\substack{-2\mu}})(K^{\substack{10\\1}}_{\substack{0}} + K^{\substack{01\\1}}_{\substack{0}}),
\end{aligned} \tag{27}
$$

where $j = 0, 1, 2$, $|\mu| \leq j$, and provided $i\sigma = -\frac{1}{\sqrt{3}}\gamma_1 + \sqrt{\frac{2}{3}}\gamma_2 - 2i$, $\delta_1 = \gamma_3$, and $\delta_2 = \frac{1}{\sqrt{3}}\gamma_1 + \frac{1}{\sqrt{6}}\gamma_2 - 2i$. Derivation of (22) is now obvious for $\gamma_2 = \gamma_3 = 0$. In a parallel to the $sl(3,\mathbb{R})$ algebra case, the generalized Gell-Mann formula for the $sl(4,\mathbb{R})$ algebra case holds for all representations, irrespectively of the $su(4)$ representations multiplicity.

It is now straightforward to write down the generalized Gell-Mann formula expressions for the $su(3)$ and $su(4)$ algebras, thus obtaining the generators of the $SU(3)/SO(3)$ and $SU(4)/SO(4)$ factor groups.

$su(3)$: $T^{su(3)}_\mu = iT_\mu$, $\mu = 0, \pm1, \pm2$, where T_μ is given by (25).

$su(4)$: $T^{su(4)}_{\mu_1\mu_2} = iT_{\mu_1\mu_2}$, $\mu_1, \mu_2 = 0, \pm 1$, i.e. $T^{su(4)}_{j\mu} = iT_{j\mu}$, $j = 0, 1, 2$, $\mu \le |j|$, where $T_{\mu_1\mu_2}$ and $T_{j\mu}$ are given by (26) and (27), respectively.

5 Generalized Gell-Mann formula in the $sl(5, \mathbb{R})$ case

We have shown above that the original Gell-Mann formula expressions for the $sl(n, \mathbb{R})$ and $su(n)$ algebras do not satisfy the $[T, T]$ commutation relations (3) on a pure algebraic level. However, by setting the Gell-Mann formula existence question into a group representation framework, it is possible to restrict representation spaces and thus achieve a closure of the $[T, T]$ commutator (T being given by the Gell-Mann formula expression). Moreover, we have shown, by extracting information from the known results about the $sl(n, \mathbb{R})$, $n = 3, 4$ representations, that there exist a generalization of the Gell-Mann formula for $sl(n, \mathbb{R})$, $n = 3, 4$, which is valid for all representation spaces. An important role, in that process, was played by the K operator (23).

In the following, we make use of the Gell-Mann formula generalization for $sl(n, \mathbb{R})$, $n = 3, 4$, and a peculiarity of the $so(5)$ algebra representation labels to follow the $so(4) = so(3) \oplus so(3)$ labeling features.

Let us recall first some basic $so(5)$ algebra representation notions. The $so(5)$ algebra is of rang two, and its irreducible representations are labeled by a pair of labels $(\overline{J}_1, \overline{J}_2)$, resembling the $so(4)$ labeling. The complete labeling of the representation space vectors can be achieved by making use of the subalgebra chain: $so(5) \supset so(4) = so(3) \oplus so(3) \supset so(2) \oplus so(2)$. The basis of the $so(5)$ algebra representation space can be taken as in [19, 20]:

$$\left\{ \left| \begin{array}{cc} \overline{J}_1 & \overline{J}_2 \\ J_1 & J_2 \\ m_1 & m_2 \end{array} \right\rangle, \quad \overline{J}_i = 0, \frac{1}{2}, \ldots; \quad \overline{J}_1 \ge \overline{J}_2; \quad |m_i| \le J_i, \quad i = 1, 2 \right\}. \tag{28}$$

The admissible values of J_1 and J_2, within an irreducible representation $(\overline{J}_1, \overline{J}_2)$ are given in [21]. Now, the basis of the $so(5)$ algebra, i.e. the $Spin(5)$ group, representation space vectors (10) is given as follows:

$$\left\{ \left| \begin{array}{cccc} \overline{J}_1 & \overline{J}_2 & & \\ K_1 & K_2 & J_1 & J_2 \\ k_1 & k_2 & m_1 & m_2 \end{array} \right\rangle \right\}. \tag{29}$$

The ten $so(5)$ algebra operators, generating the adjoint representation of $Spin(5)$, transform, in notation (28), under the representation $(\overline{1},\overline{0})$. Their $so(4)$ subalgebra representation content is: $(\overline{1},\overline{0}) \to (1,0) \oplus (\frac{1}{2},\frac{1}{2}) \oplus (0,1)$. The shear operators transform under the 14-dimensional $so(5)$ irreducible representation $(\overline{1},\overline{1})$ of $so(5)$ which contains $(1,1)$, $(\frac{1}{2},\frac{1}{2})$ and $(0,0)$ representation upon reduction to $so(4)$:

$$\left\{ T_{\substack{j_1 j_2\\ \mu_1\mu_2}} \right\} = \left\{ T_{\substack{1\,1\\ \mu_1\mu_2}}, T_{\substack{\frac{1}{2}\frac{1}{2}\\ \mu_1\mu_2}}, T_{\substack{00\\ 00}} \right\}.$$

The original Gell-Mann formula is not applicable, again, in the whole space spanned by (29), but only in the symmetric spaces $Spin(5)/Spin(4)$ and $Spin(5)/(Spin(3) \otimes Spin(2))$, with appropriate choices of the $SO(5)_K$ labels for the contracted operators U. Neither representations with multiplicity, nor spinorial representations of the $sl(5,\mathbb{R})$ algebra can be obtained in this way.

We made an educated guess, based on the structure of the generalized Gell-Mann formula for the $sl(4,\mathbb{R})$ algebra, when adding possible terms to the generalized Gell-Mann formula in the $sl(5,\mathbb{R})$ case. We omit the terms proportional to $D^{\overline{1}\,\overline{1}}_{\substack{\frac{1}{2}\frac{1}{2}\, j_1 j_2\\ k_1 k_2 \mu_1\mu_2}}$ altogether, as well as the terms proportional to $D^{\overline{11}}_{\substack{11\, j_1 j_2\\ 01\, \mu_1\mu_2}}$, $D^{\overline{11}}_{\substack{11\, j_1 j_2\\ 10\, \mu_1\mu_2}}$, $D^{\overline{11}}_{\substack{11\, j_1 j_2\\ 0-1\, \mu_1\mu_2}}$, and $D^{\overline{1}\,\overline{1}}_{\substack{1\,1\, j_1 j_2\\ -1\,0\, \mu_1\mu_2}}$.

The $[T,T] \subset M$ commutation relations condition (3), together with a knowledge of the $so(5)$ Clebsch-Gordan coefficients for $(\overline{J_1},\overline{J_2}) = (1,1)$ [20], finally yields the sought for generalized Gell-Mann formula expression for the $sl(5,\mathbb{R})$ algebra shear operators:

$$\begin{aligned} T_{\substack{j_1 j_2\\ \mu_1\mu_2}} = {} & \sigma_1 D^{\overline{11}}_{\substack{00\, j_1 j_2\\ 00\, \mu_1\mu_2}} + i\sqrt{\tfrac{1}{5}}\left[C_2(so(5)), D^{\overline{11}}_{\substack{00\, j_1 j_2\\ 00\, \mu_1\mu_2}}\right] \\ & + i\Bigg(\sigma_2 D^{\overline{11}}_{\substack{11\, j_1 j_2\\ 00\, \mu_1\mu_2}} + \tfrac{1}{2}\left[C_2(so(4)_K), D^{\overline{11}}_{\substack{11\, j_1 j_2\\ 00\, \mu_1\mu_2}}\right] \\ & - D^{\overline{1}\,\overline{1}}_{\substack{1\,1\, j_1 j_2\\ 1\,-1\, \mu_1\mu_2}} \left(\delta_1 + K^{\overline{10}}_{\substack{10\\ 00}} - K^{\overline{10}}_{\substack{01\\ 00}}\right) - D^{\overline{1}\,\overline{1}}_{\substack{1\,1\, j_1 j_2\\ -1\,1\, \mu_1\mu_2}} \left(\delta_1 - K^{\overline{10}}_{\substack{10\\ 00}} + K^{\overline{10}}_{\substack{01\\ 00}}\right) \\ & + D^{\overline{11}}_{\substack{11\, j_1 j_2\\ 11\, \mu_1\mu_2}} \left(\delta_2 + K^{\overline{10}}_{\substack{10\\ 00}} + K^{\overline{10}}_{\substack{01\\ 00}}\right) + D^{\overline{1}\,\overline{1}}_{\substack{1\,1\, j_1 j_2\\ -1\,-1\, \mu_1\mu_2}} \left(\delta_2 - K^{\overline{10}}_{\substack{10\\ 00}} - K^{\overline{10}}_{\substack{01\\ 00}}\right) \Bigg), \end{aligned} \tag{30}$$

where $j_i = 0, \frac{1}{2}, 1$, $|\mu_i| \leq j_i$, $i = 1,2$, the representation labels $\sigma_1, \sigma_2, \delta_1$ and δ_2 are arbitrary (complex) parameters (four real are independent), and $C_2(so(4)_K)$ denotes the quadratic Casimir operator of the left action $so(4)_K$

algebra. Naturally, the same result (30) is obtained, when searching for the generalized Gell-Mann formula expression, by starting with all possible additional terms proportional to $D^{\begin{smallmatrix} \overline{1} & \overline{1} & & \\ K_1 & K_2 & j_1 & j_2 \\ k_1 & k_2 & \mu_1 & \mu_2 \end{smallmatrix}}$ -functions and demanding (3), though by a much more tedious calculation.

The $su(5)$ algebra elements, as given by the generalized Gell-Mann formula, are the M operators (11), generating the $SO(5)$ subgroup of the $SU(5)$ group, and the iT operators (30), generating the $SU(5)/SO(5)$ factor group.

Contrary to the generalized Gell-Mann formula for the $sl(n, \mathbb{R})$, $n = 3, 4$ algebras, where we started from the known matrix elements of the shear operators, here in the case of the $sl(5, \mathbb{R})$ algebra, we are in a position to obtain, for the first time, the shear operators matrix elements for a generic representation space starting from the generalized Gell-Mann formula expression (30).

The matrix elements of the $sl(5, \mathbb{R})$ shear (noncompact) operators read:

$$\begin{aligned}
&\left\langle \begin{smallmatrix} \overline{J}'_1 & \overline{J}'_2 & & \\ K'_1 & K'_2 & J'_1 & J'_2 \\ k'_1 & k'_2 & m'_1 & m'_2 \end{smallmatrix} \middle| T_{\begin{smallmatrix} j_1 j_2 \\ \mu_1 \mu_2 \end{smallmatrix}} \middle| \begin{smallmatrix} \overline{J}_1 & \overline{J}_2 & & \\ K_1 & K_2 & J_1 & J_2 \\ k_1 & k_2 & m_1 & m_2 \end{smallmatrix} \right\rangle = \sqrt{\frac{dim(\overline{J}_1, \overline{J}_2)}{dim(\overline{J}'_1, \overline{J}'_2)}} C^{\begin{smallmatrix} \overline{J}_1 \overline{J}_2 & \overline{1}\ \overline{1} & \overline{J}'_1 \overline{J}'_2 \\ J_1\ J_2 & j_1 j_2 & J'_1\ J'_2 \\ m_1 m_2 & \mu_1 \mu_2 & m'_1 m'_2 \end{smallmatrix}} \\
&\times \Bigg(\Big(\sigma_1 + i\sqrt{\tfrac{4}{5}}(\overline{J}'_1(\overline{J}'_1+2) + \overline{J}'_2(\overline{J}'_2+1) - \overline{J}_1(\overline{J}_1+2) - \overline{J}_2(\overline{J}_2+1))\Big) C^{\begin{smallmatrix} \overline{J}_1 \overline{J}_2 & \overline{1}\overline{1} & \overline{J}'_1 \overline{J}'_2 \\ K_1 K_2 & 00 & K'_1 K'_2 \\ k_1\ k_2 & 00 & k'_1\ k'_2 \end{smallmatrix}} \\
&\quad + i\Big(\sigma_2 + K'_1(K'_1+1) + K'_2(K'_2+1) - K_1(K_1+1) - K_2(K_2+1)\Big) C^{\begin{smallmatrix} \overline{J}_1 \overline{J}_2 & \overline{1}\overline{1} & \overline{J}'_1 \overline{J}'_2 \\ K_1 K_2 & 11 & K'_1 K'_2 \\ k_1\ k_2 & 00 & k'_1\ k'_2 \end{smallmatrix}} \\
&\quad - i(\delta_1 + k_1 - k_2) C^{\begin{smallmatrix} \overline{J}_1 \overline{J}_2 & \overline{1}\overline{1} & \overline{J}'_1 \overline{J}'_2 \\ K_1 K_2 & 11 & K'_1 K'_2 \\ k_1\ k_2 & 1 -1 & k'_1\ k'_2 \end{smallmatrix}} - i(\delta_1 - k_1 + k_2) C^{\begin{smallmatrix} \overline{J}_1 \overline{J}_2 & \overline{1}\ \overline{1} & \overline{J}'_1 \overline{J}'_2 \\ K_1 K_2 & 1\ 1 & K'_1 K'_2 \\ k_1\ k_2 & -1 1 & k'_1\ k'_2 \end{smallmatrix}} \\
&\quad + i(\delta_2 + k_1 + k_2) C^{\begin{smallmatrix} \overline{J}_1 \overline{J}_2 & \overline{1}\overline{1} & \overline{J}'_1 \overline{J}'_2 \\ K_1 K_2 & 11 & K'_1 K'_2 \\ k_1\ k_2 & 11 & k'_1\ k'_2 \end{smallmatrix}} + i(\delta_2 - k_1 - k_2) C^{\begin{smallmatrix} \overline{J}_1 \overline{J}_2 & \overline{1}\ \overline{1} & \overline{J}'_1 \overline{J}'_2 \\ K_1 K_2 & 1\ 1 & K'_1 K'_2 \\ k_1\ k_2 & -1 -1 & k'_1\ k'_2 \end{smallmatrix}} \Bigg),
\end{aligned} \tag{31}$$

where $dim(\overline{J}_1, \overline{J}_2) = (2\overline{J}_1 - 2\overline{J}_2 + 1)(2\overline{J}_1 + 2\overline{J}_2 + 3)(2\overline{J}_1 + 2)(2\overline{J}_2 + 1)/6$ is the dimension of the $so(5)$ irreducible representation characterized by $(\overline{J}_1, \overline{J}_2)$ [21].

To sum up, the matrix elements of the (noncompact) shear operators T (31), together with the known matrix elements of the (compact) $so(n)$ operators M (11), define an action of the $sl(5, \mathbb{R})$ algebra on the basis vectors (29) of representation spaces of the maximal compact subgroup $Spin(5)$ of the $\overline{SL}(5, \mathbb{R})$ group. This result is general due to a Corollary of Harish-Chandra [17] that explicitly applies to the case of the $sl(5, \mathbb{R})$ algebra.

6 Conclusion

The Gell-Mann formula, beyond the case of (pseudo) orthogonal algebras, is not valid in general as a pure algebraic expression. Its applicability can be broaden, by utilizing it in certain cases, provided some Lie algebra representation conditions are met. As for the $sl(n,\mathbb{R})$ algebras, contracted w.r.t. their $so(n)$ subalgebras, the algebraic expression of the Gel-Mann formula matters generally for the multiplicity free representations only. It was demonstrated in this work that one can generalize the Gell-Mann formula for the $sl(n,\mathbb{R})$, $n = 3, 4, 5$ algebras to be valid for a generic representation space. A brief account required by a description of the $sl(n,\mathbb{R})$, $n = 3, 4, 5$ representation spaces, that heavily depends on the corresponding group topology properties and the $so(n)$ subgroup multiplicity, is given. In the $sl(3,\mathbb{R})$ and $sl(4,\mathbb{R})$ cases we inferred, starting from a suitable existing expressions of the algebra operators representations with non-trivial multiplicity, a generic generalized Gell-Mann formulas. These formulas offer a new starting point for mathematical physics investigations, as well as for various physical applications. By analyzing the structure of the Gell-Mann formula for the $sl(n,\mathbb{R})$, $n = 3, 4, 5$ cases, and making use of the specific features of the $so(5)$ algebra representations, we obtained the generalized Gell-Mann formula for the $sl(5,\mathbb{R})$ and $su(5)$ cases. Note that this formula, that is valid for all representation Hilbert spaces, is characterized precisely by a right number (algebra rank) of the representation labels. As a first and most precious application, based on this generalized Gell-Mann formula, we obtained for the first time a closed form of the generic expressions of all matrix elements of the $sl(5,\mathbb{R})$ noncompact generators. A distinct feature of our generalized Gell-Mann formula approach is that the resulting expression goes beyond the standard notion of a deformation of the contracted algebra, as it depends on additional operators not belonging, however directly related, to the contracted algebra. Due to this fact, our generalization of the Gell-Mann formula is remarkable simple (compared to complicated polynomial expressions appearing in some other approaches to generalize the Gell-Mann formula), nevertheless establishing a direct relation between representations of the contracted and original algebras.

7 Acknowledgments

This work was supported in part by MNTR, Project-141036. One of us, I.S., would like to acknowledge hospitality and useful discussions at the Institute for Nuclear Research and Nuclear Energy in Sofia (Bulgaria) during his visit as early stage researcher supported by the FP6 Marie Curie Research Training Network "Forces-Universe" MRTN-CT-2004-005104.

References

[1] Inönü E and Wigner E P 1953 *Proc. Nat. Acad. Sci.* **39** 510

[2] Mackey G W 1968 *Induced Representations of Groups and Quantum Mechanics* (New York: Benjamin)

[3] Hermann R 1965 *Lie Groups for Physicists* (New York: W. A. Benjamin Inc)

[4] Hazewinkel M ed. 1997 *Encyclopaedia of Mathematics, Supplement I* (Springer)

[5] Salom I and Šijački Dj 2009 *SFIN* **XXII A**, "$\overline{SL}(5, R)$ fields in gravity and brane physics".

[6] Hermann R 1966 *Commun. Math. Phys.* **2** 155

[7] Berendt G 1967 *Acta Phys. Austriaca* **25** 207

[8] Sankaranarayanan A 1965 *Nuovo Cimento* **38** 1441

[9] Weimar E 1972 *Lettere Al Nuovo Cimento* **4** 2

[10] Štoviček P 1988 *J. Math. Phys.* **29** 1300

[11] N. Mukunda, *J. Math. Phys.* **10** (1969) 897

[12] Šijački Dj 2008 *Class. Quant. Grav.* **25** 065009

[13] Ne'eman Y and Šijački Dj 1987 *Int. J. Mod. Phys. A* **2** 1655

[14] Šijački Dj 1990 *J. Math. Phys.* **31** 1872

[15] Šijački Dj 1975 *Jour. Math. Phys.* **16** 298

[16] Sijacki Dj 1988 in *Spinors in Physics and Geometry*, A. Trautman and G. Furlan eds. (World Scientific Pub.) 191

[17] Harish-Chandra 1951 *Proc. Nat. Acad. Sci.* **37** 170, 362, 366, 691

[18] Salom I and Šijački Dj 2009 *SFIN* **XXII A**, "Generalization of Gell-Mann formula for $sl(3,\mathbb{R})$ and $sl(4,\mathbb{R})$ algebra".

[19] Hecht K T 1965 *Nucl. Phys.* **63** 177

[20] Salom I and Šijački Dj, "$\overline{SO}(5)$ Clebsch-Gordan coefficients involving the 14-dimensional representation", to be published

[21] Kemmer N, Pursey D L and Williams S A 1968 *J. Math. Phys.* bf 9 1224